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SUBJECT: Resolution Beyond the "Diffraction Limit" in Space Astronomy - Case 105-3

DATE: March 28, 1969

FROM: W. D. Grobman

ABSTRACT

In principle it is possible to reconstruct an astronomical object with infinitely fine resolution from the measured image intensity provided by an ideal, noisefree telescope of finite aperure. In the presence of noise, this reconstruction can only be carried past the "Rayleigh limit" to a certain point. There are important implications of these facts for space astronomy, which make worthwhile the study of theories of artificially increasing resolution. We discuss the application of these techniques to real systems and conclude that the small amount of unavoidable noise in actual telescopes precludes extension of their resolution significantly beyond the Rayleigh limit.

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MEMORANDUM FOR FILE

I. Introduction:

The achievement of "diffraction-limited" telescope performance is an exciting possibility for space astronomy and is one of its primary justifications. The angular resolution of a space telescope is commonly stated to be about λ/D where λ is the wavelength of light and D is the telescope aperture. However, the theory of linear filters in electrical engineering has shown that in principle it is possible to achieve an effective angular resolution much finer than λ/D. Knowing the optical characteristics of a telescope and certain general a priori information about an astronomical object, we can recreate every detail of the original object in the absence of noise.(1) Of course no actual system is noise-free, so the theory of object reconstruction can only be used in practice to a limited extent. Nevertheless, if we could only double the resolution of a space telescope - e.g., achieve the resolution performance of a two meter telescope with a one meter telescope, the implications for the space program would be great. These facts justify this short study, even though we will find that for any actual telescope only a small resolution increase can be achieved by artificially enhancing the images it yields.

The present paper discusses two particular schemes for enhancing the resolution of space telescopes. The following section will discuss the procedures which theoretically can be used to enhance the resolution of an optical image. We then treat the effects of noise on the ability to increase resolution. A discussion of the significance of these results for the kinds of observation of most interest to space astronomy is given in Section IV, and the reader who is not interested in the details of the theory may skip to that section now.

II. The Concept of Object Restoration:

This section discusses in a general way the concepts involved in the theory of reconstructing a high resolution object from a low resolution image. The following section is less tutorial and considers the quantitative implications of this theory for real telescopes.

In the following analysis we will for simplicity restrict ourselves to one-dimensional images and objects. The results are easily extended to actual two-dimensional objects because the two dimensions are imaged independently since the theory is linear. The telescopes used will have square apertures, which gives results differing only slightly from those for a round aperture. These assumptions simplify the analysis significantly, but the results are still applicable to real telescopes.

The object intensity in the object plane is denoted by $s(x)$, and the image intensity (in the absence of noise) is $r(x)$. Units in the object and image planes are so chosen that the same coordinate, x, describes corresponding points in the two planes. The effect of a telescope is to spread a point source of light in the object plane into a region in the image plane. That is, if $s(x) = \delta(x-u)$, then $r(x) = h(x-u)$, where $\delta(x)$ is the Dirac delta function (unit impulse function at $x = u$) and $h(x)$ is the <u>point spread function</u> of the telescope and describes diffraction effects arising from the finite wavelength of light. The image of a general object is determined from

$$r(x) = \int_{-\infty}^{\infty} du\; h(x-u)s(u) \tag{1}$$

Since this is a convolution, the fourier transforms of r, h, and s are related by

$$R(f) = H(f)S(f) \tag{2}$$

where f is spatial frequency (e.g., cycles/mm) and where

$$R(f) = \int_{-\infty}^{\infty} dx\; e^{2\pi i f x}\, r(x) \tag{3}$$

and similarly for $H(f)$ and $S(f)$. $H(f)$ is known as the <u>Optical Transfer Function</u> of the telescope and tells how the (complex) amplitude of the object at each spatial frequency is affected by the telescope to give the amplitude of the corresponding frequency component of the image. The modulus of the optical transfer function is known as the Modulation Transfer Function.

H(f) has the property that it is zero beyond some critical spatial frequency f_o. This inability of a telescope to pass high spatial frequencies means that it cannot physically image details of an object with a resolution greater than some definite value which corresponds to the Rayleigh limit if "resolution" is properly defined. However, we can show that even though the image intensity r(x) does not contain frequencies greater than f_o, it can be mathematically manipulated to yield s(x), which contains all frequencies, provided that s(x) is of finite extent in space.(1) This limitation is met by the objects of interest in astronomy (e.g. galaxies, double star systems, etc.).

The general idea in reconstructing s(x) is the following. If s(x) is a finite object ($s(x)=0$ for $|x|>L$), then S(f) is an analytic function of f. This fact is easily proven (for example, see Reference 1). The measured r(x) can be used to form R(f), and h(x) or H(f) are known. Therefore, S(f) can be found for $f<f_o$ from equation (2). Once S(f) for $f<f_o$ is known, then S(f) for $f>f_o$ can be found by analytically continuing S(f). That is, since S(f) is analytic, its value in the restricted frequency domain passed by the telescope determines its value at all frequencies. This object reconstruction process is schematically represented in Figure 1. From this figure we can see the dependence of the results on the assumption that there is no noise. If the S(f) ($f<f_o$) that we produce differs even slightly from its true value, then as we analytically extend S(f) to higher values of f, the error can increase very rapidly.

This discussion has shown why it is possible in principle to reconstruct an object at high frequencies. To quantitatively analyze the effects of noise on this reconstruction, it is convenient to use a different reconstruction scheme, one which has been studied extensively.(2) It is based not on the frequency space relation between S(f) and R(f) (equation (2)), but rather on the integral relation between s(x) and r(x) (equation (1)). Equation (1) is an integral equation relating the measured intensity r(x) to the unknown object intensity s(x) via a known kernel h(x-u). Since $s(x)=0$ for $|x|>L$, we are trying to solve

$$r(x) = \int_{-L}^{L} dx\ h\ (x-u)s(x). \tag{4}$$

The solution of (4) is given in terms of the eigenfunctions $\phi_i(x)$ and engenvalues λ_i of the integral operator in (4):

$$\int_{-L}^{L} dx\ h(x-u)\phi_i(x) = \lambda_i\phi_i(x). \tag{5}$$

From equations(5) and (4) it is easy to show that s(x) is given in terms of r(x) by

$$s(x) = \sum_{i=o}^{\infty} \frac{1}{\lambda_i} <\phi_i,r>\phi_i(x) \tag{6}$$

where we have assumed that the $\phi_i(x)$ have been made ortho-normal and where

$$<\phi_i,r> \equiv \int_{-L}^{L} dx\ \phi_i(x)r(x)\ . \tag{7}$$

Another more transparent form of (6) is

$$\lambda_i<\phi_i,s> = <\phi_i,r> \tag{8}$$

which can be simply interpreted. Equation (8) states that the projection of the image r(x) onto the basis function $\phi_i(x)$ ("the i^{th} component of r(x)") is simply λ_i times the projection of s(x) onto $\phi_i(x)$. The greater i is, the more nodes there are in $\phi_i(x)$ and the smaller λ_i is, so that "high frequency" components of the object are multiplied by a small number to give the high frequency content of the image. We can now show in a crude way why noise in the image plane can be so damaging to the reconstruction. If we measure an intensity r(x)+n(x) in the image plane, where n(x) is noise added to the image r(x), and perform the operation shown in equation (6), then we obtain an incorrect reconstruction of the object intensity, denoted $\bar{s}(x)$:

$$\bar{s}(x) = \sum_{i=o}^{\infty} \frac{1}{\lambda_i}[<\phi_i,r>+<\phi_i,n>]\phi_i(x). \tag{9}$$

The error in the reconstruction, e(x), is the difference between $\bar{s}(x)$ and s(x):

$$e(x) \equiv \bar{s}(x) - s(x) = \sum_{i=o}^{\infty} \frac{1}{\lambda_i}<n,\phi_i>\phi_i(x), \tag{10}$$

or equivalently

$$<e,\phi_i> = \frac{1}{\lambda_i}<n,\phi_i> . \tag{11}$$

Equation (11) displays very clearly the fact that the error at high spatial frequencies is given by the noise at these frequencies multiplied by a large number $\frac{1}{\lambda_i}$ - the larger i is the smaller λ_i is and it approaches zero extremely fast as i increases.

This discussion shows that as i increases to infinity in the sum in equations (6), (9), and (10), the effect of the error increases without bounds unless $<n,\phi_i> \to 0$ very quickly for large i. Thus for a real system the series in (9) should be terminated at a value of i, denoted N, which is high enough to give a good representation of s(x), but which is low enough that the noise does not completely submerge the reconstructed object. If the value of N obtained in this corresponds to a function ϕ_N which has higher frequency components than those corresponding to the Rayleigh criterion, then it is possible to go beyond "diffraction-limited" performance.

III. <u>The Quantitative Effect of Noise on Object Reconstruction</u>:

The preceeding section has given the conceptual foundations of the object reconstruction problem. We now discuss the effect of noise in a quantitative way in order to determine if it is practical to extend the resolution of a space telescope beyond the Rayleigh limit. To do this, we must treat specific object reconstruction techniques, as the quantitative results might depend on the method used. We will consider two particular techniques. One of these (method A) is based on an expansion in the eigenfunctions of the integral operator based on the point spread function - an expansion in the $\phi_i(x)$ of the preceeding section. The other technique is based on analytically extending $S(f)$, $f<f_o$, into the $f>f_o$ domain (method B).

Method A:

The general ideas of this method are contained in equations (5) and (6) in the preceeding section. If the particular $h(x)$ characterizing a space telescope is used in (5), then the solutions of (5), along with the measured values of $r(x)$, determine $s(x)$ from equation (6). For the square aperture telescope that we use,

$$h(x) = 2f_o \operatorname{sinc} 2f_o x \quad \text{(Reference 4)} \tag{12}$$

where

$$\operatorname{sinc} u \equiv \frac{\sin \pi u}{\pi u} \tag{13}$$

The cutoff frequency f_o is given in terms of the telescope aperture D, focal length F, and the wavelength of light by

$$f_o = \frac{D}{2\lambda F} \ . \tag{14}$$

The eigenfunctions and eigenvalues of (5) with the kernal given in (12) have been studied in detail in the literature(3), and their specific form need not be given here. If these eigenfunctions are used in (6), and if the sum in (6) only extends from $i = 0$ to $i = N$, then the mean squared error in $s(x)$ due to noise can be calculated, given the noise statistics. This has been done by Rushforth and Harris(4) for different types of noise. For white noise in the image which has had its frequency components greater than f_o discarded, the mean squared error in the reconstructed object $\bar{s}(x)$ is proportional to $\sum_{i=0}^{N} \frac{1}{\lambda_i}$. This sum is shown in Figure 2 as a function of $\frac{N}{2Lf_o}$ and the parameter $C = 2\pi Lf_o$. The first variable is simply the ratio (resolution in the reconstructed object)/(Rayleigh criterion resolution). The parameter C may be interpreted as the number of Rayleigh resolution distance elements spanned by the object (e.g., a galaxy whose angular size is five times λ/D would have $C \approx 5$). Note the precipitous rise of the sum over λ_i^{-1} as the Rayleigh criterion resolution is surpassed. The rise is so fast that <u>any reasonable noise in the image will not permit reconstruction beyond some definite resolution determined by the value of C</u>. This limiting resolution is given as a function of C in Figure 3 for two signal/noise ratios. These curves were obtained by determining from Figure 2 the resolution at which the noise power is increased to the extent that it equals the signal power. The dashed line shows the resolution achievable, given an *a priori* object size, without using the methods described here. Note the insensitivity of the resolution achievable to large changes in the signal/noise ratio. The significance of this result will be discussed in the next section, but it is immediately obvious that this method offers little possibility of increasing the resolution of an actual system.

Method B

This method, described in detail in Reference 5, uses the Whittaker-Shannon sampling theorem of information theory to analytically continue the spatial frequency components of $s(x)$ from their values for $f<f_o$ to the region $f>f_o$. The effect of noise has not to our knowledge been analytically studied for this method. We have modeled a telescope and this reconstruction method on a computer[6], and have empirically determined at what resolution, as a function of the parameter C, this method breaks down due to noise. The results show that this method is subject to the same limitations as Method A. Further discussion of this method will be documented separately.

We conclude from this analysis that, independent of the reconstruction method used, there are fundamental limitations to improving telescope resolution beyond a rather well-defined point in the presence of any reasonable amount of noise. This point may, however, be less than the usually stated diffraction criterion (but not much less for objects of astronomical interest). The important parameter which determines the resolution that can be obtained is

$$C \equiv 2\pi \, Lf_o \tag{15}$$

which, as noted before, is the number of Rayleigh criterion resolution elements spanned by the object (i.e., how many times larger than λ/D it is). We now go on to consider the relation of the needs of astronomy to the results of this analysis.

IV. Implications for Space Astronomy:

The preceeding sections have shown that only slight improvements in resolution can be achieved in practice by artificial enhancement of images. To know whether these slight improvements are significant for space astronomy we must consider the types of astronomical observations which have been identified as benefitting from high resolution[7]. High resolution is scientifically valuable in four major areas[7] which we now discuss individually:

1. Detection of Faint Point Sources:

This is achieved with a high resolution telescope because the physical size of the image of a point source in the focal plane is small. Such concentration of the light from an object increases its contrast with the background light from the sky and so permits fainter sources to be detected.[8] Calculations show that a 120 inch, diffraction-limited telescope could detect 29th magnitude stars, while the best that can be done from the earth is about 24th magnitude. This represents the detection of sources 100 times fainter than can be detected from the earth. Obviously this is a physical, light gathering process, and the analytical techniques that we have been discussing are not relevant to this problem.

2. Study of the Large Scale Structure of the Universe:

This is achieved by measuring the velocities and distribution in space of galaxies more distant than those observed from the earth's surface. The velocities of these faint sources can be measured spectroscopically only because their light is concentrated into a small region of the focal plane by a high resolution telescope. Thus this measurement cannot benefit from artificial enhancement. The distance of galaxies is measured in two ways. The most accurate and the one used for nearby galaxies involves resolving certain types of stars of known luminosity and measuring their apparent brightness. To resolve small scale details of a galaxy, it must occupy an area in the focal plane covering many Rayleigh resolution cells. For such an object the parameter C defined in the previous section is large and from Figure 3 we see that for large C only the Rayleigh resolution can be achieved, so that artificial resolution enhancement techniques are of no use here. The angular size of more distant galaxies provides a crude measure of their distance or of the curvature of the universe. If the angular size of such a galaxy is greater than λ/D, it can be measured directly. If it is smaller, we cannot apply the theory of the preceeding sections unless we have an a priori bound on the size of the object. We do not have such a bound since it is the size itself that we are trying to measure, so resolution enhancement techniques are of no use here.

3. Structure and Evolution of Individual Galaxies:

This measurement again involves a large value for the parameter C and so cannot take advantage of resolution enhancement techniques.

4. Planetary Surface and Atmosphere Mapping:

The comments just made apply to this case also.

In spite of the poor possibilities for applying resolution enhancement techniques to space astronomy, we want to emphasize that certain types of artificial image enhancement can be useful. We have shown that extension of our knowledge of an object to spatial frequencies greater than f_o, the "Rayleigh limit", is not a useful technique in Space Astronomy. However, frequencies less than f_o are passed by the telescope, and since we know the optical transfer function of the instrument we know quantitatively the extent to which each of these frequencies has been degraded. This implies that we can enhance the high frequencies in the image by the proper amount needed to obtain a better idea of the structure of the object than is obtained by looking at the raw data[8]. Such enhancement techniques would be especially useful for obtaining information about galactic and planetary structure. These techniques are currently of great interest and under intensive study. However, resolution beyond the Rayleigh limit of a given telescope will only be obtained by going to other, larger telescopes or, what is perhaps more practical, optical interferometers and aperture synthesis techniques.

V. Acknowledgement:

I am grateful to B. T. Caruthers of the Applications Programming Group in Department 1032 for computer modeling of method B described in Section III.

W. D. Grobman

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Attachment
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(6) Programming by B. T. Caruthers in Department 1032 is gratefully acknowledged. This method was applied in a manner which closely paralleled that used in Reference 5.

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(8) Hiltner, W. A., 1960, Astronomical Techniques, Vol. II of Stars and Stellar Systems, ed. Greenstein, J. (Univ. of Chicago Press, Chicago).

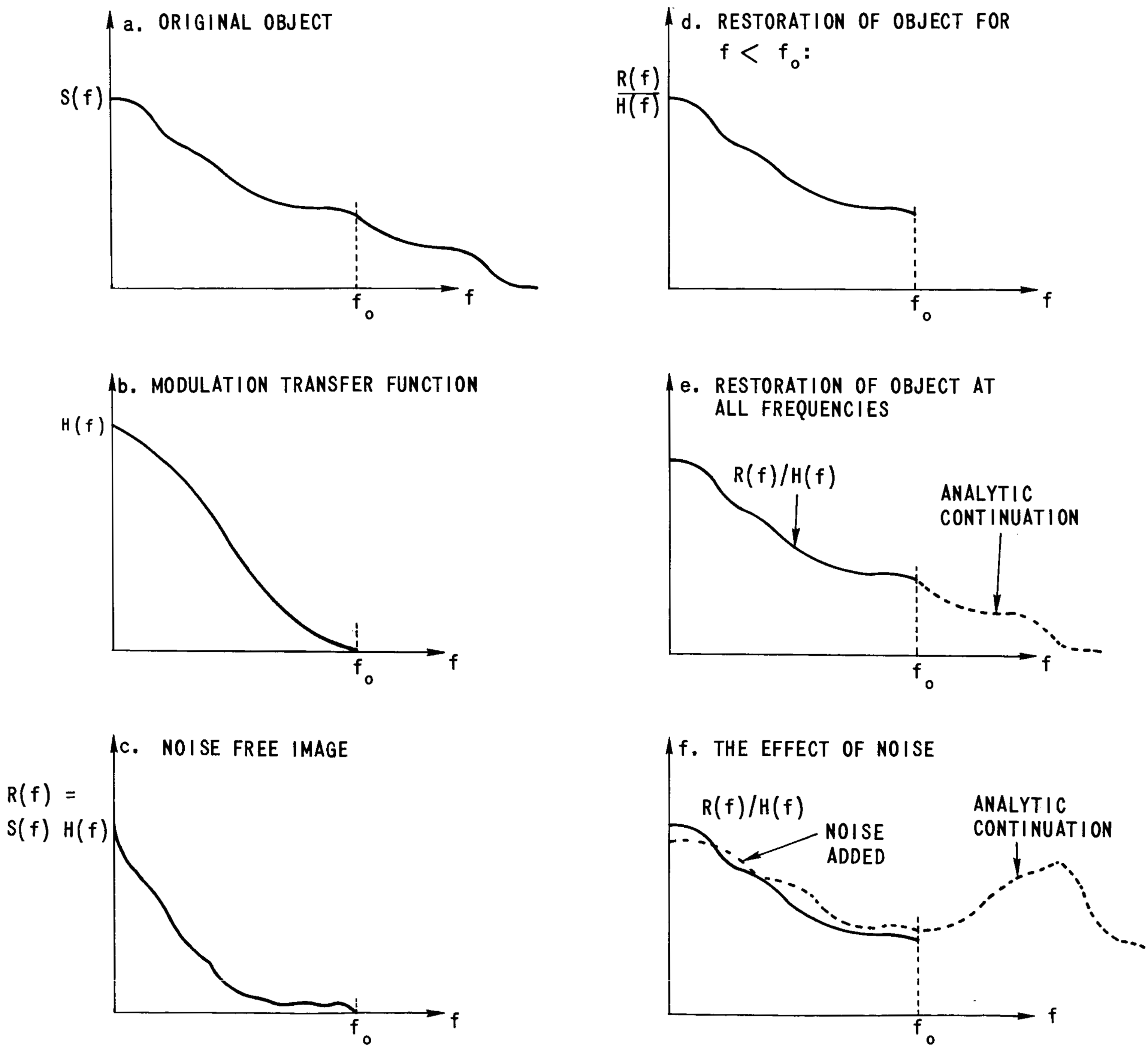
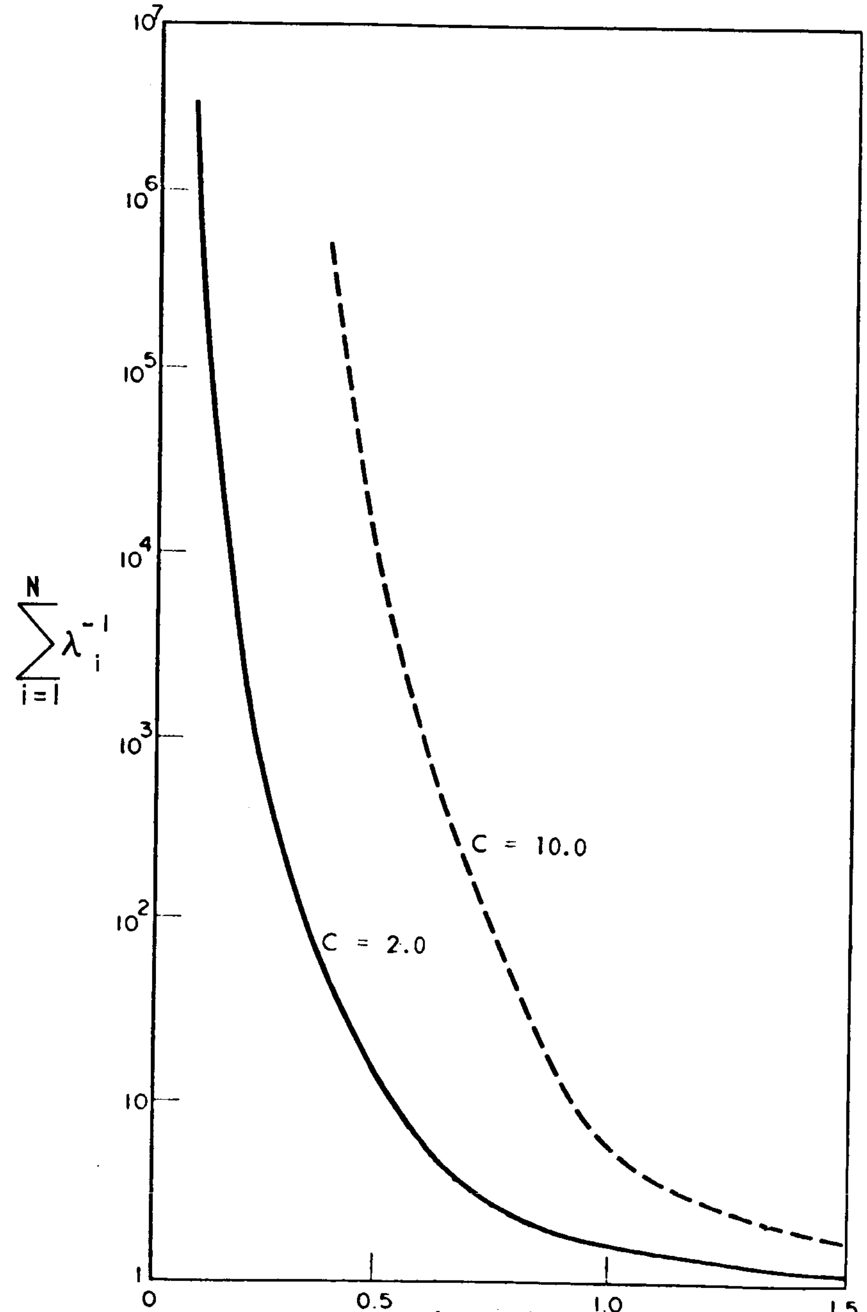


FIGURE 1 - SCHEMATIC ILLUSTRATION OF OBJECT RECONSTRUCTION PROCESS



$d_r/R = N/2Lf_o$

(R = RAYLEIGH RESOLUTION)

FIGURE 2 - SUM OVER λ_i^{-1} AS A FUNCTION OF RESOLUTION d_r

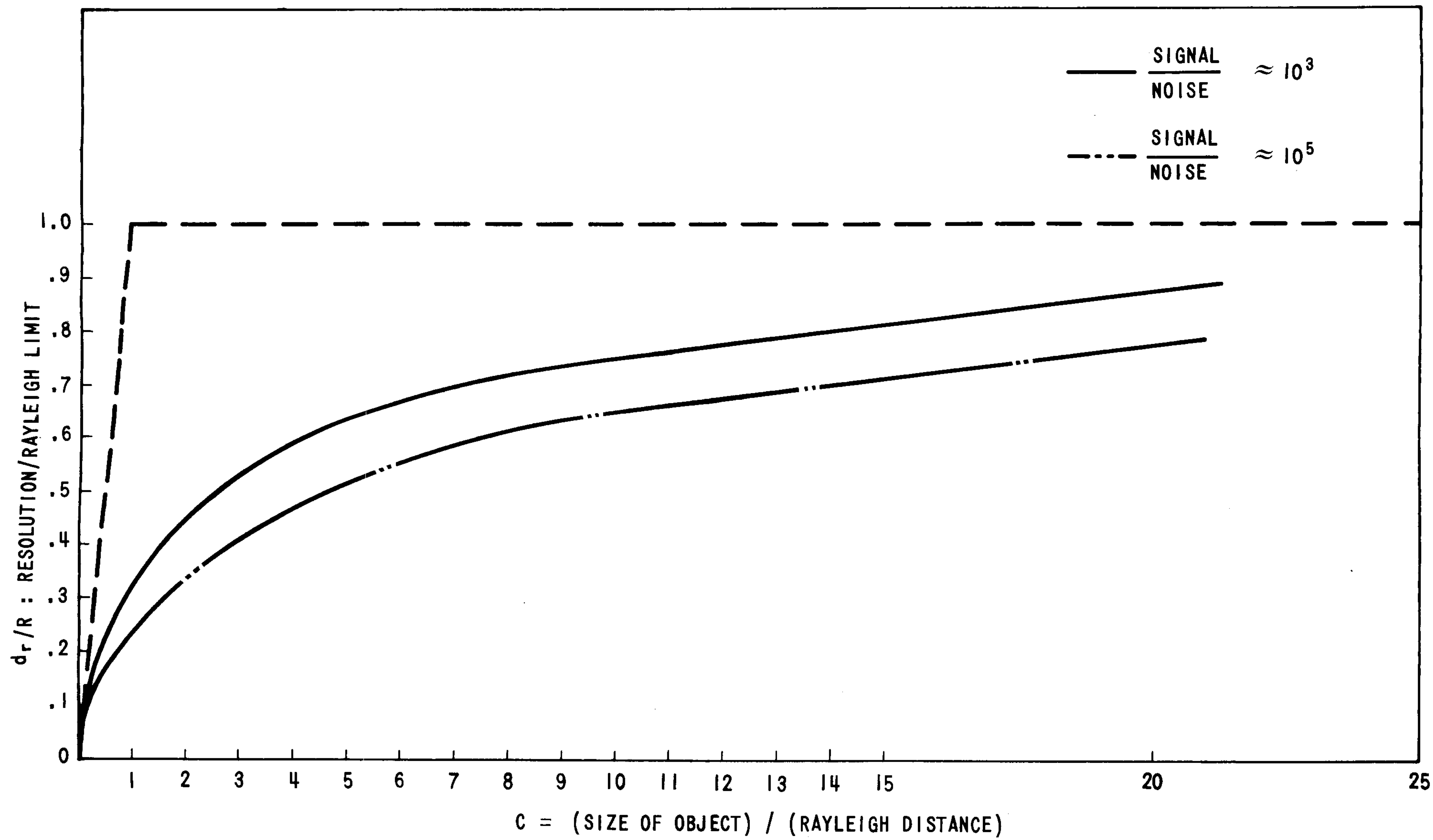


FIGURE 3 - RESOLUTION OBTAINABLE USING OBJECT RECONSTRUCTION TECHNIQUES